## Exercises

**Exercise 7.2.1** *Find the eigenvalues and eigenvectors of the matrix*

$$\begin{bmatrix} 5 & -18 & -32 \\ 0 & 5 & 4 \\ 2 & -5 & -11 \end{bmatrix}$$

*One eigenvalue is* 1*. Diagonalize if possible.*

**Exercise 7.2.2** *Find the eigenvalues and eigenvectors of the matrix*

$$\begin{bmatrix} -13 & -28 & 28 \\ 4 & 9 & -8 \\ -4 & -8 & 9 \end{bmatrix}$$

**Exercise 7.2.3** *Find the eigenvalues and eigenvectors of the matrix*

$$\begin{bmatrix} 89 & 38 & 268 \\ 14 & 2 & 40 \\ -30 & -12 & -90 \end{bmatrix}$$

*One eigenvalue is* $-3$*. Diagonalize if possible.*

**Exercise 7.2.4** *Find the eigenvalues and eigenvectors of the matrix*

$$\begin{bmatrix} 1 & 90 & 0 \\ 0 & -2 & 0 \\ 3 & 89 & -2 \end{bmatrix}$$

*One eigenvalue is* 1*. Diagonalize if possible.*

**Exercise 7.2.5** *Find the eigenvalues and eigenvectors of the matrix*

$$\begin{bmatrix} 11 & 45 & 30 \\ 10 & 26 & 20 \\ -20 & -60 & -44 \end{bmatrix}$$

*One eigenvalue is* 1*. Diagonalize if possible.*

**Exercise 7.2.6** *Find the eigenvalues and eigenvectors of the matrix*

$$\begin{bmatrix} 95 & 25 & 24 \\ -196 & -53 & -48 \\ -164 & -42 & -43 \end{bmatrix}$$

*One eigenvalue is* 5*. Diagonalize if possible.*

**Exercise 7.3.1** *Let* $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. *Diagonalize* $A$ *to find* $A^{10}$.

**Exercise 7.3.2** *Let* $A = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 5 \end{bmatrix}$. *Diagonalize* $A$ *to find* $A^{50}$.

**Exercise 7.3.3** *Let* $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -2 & 3 & 1 \end{bmatrix}$. *Diagonalize* $A$ *to find* $A^{100}$.

**Exercise 7.4.1** *Find the eigenvalues and an orthonormal basis of eigenvectors for* $A$.

$$A = \begin{bmatrix} 11 & -1 & -4 \\ -1 & 11 & -4 \\ -4 & -4 & 14 \end{bmatrix}$$

**Exercise 7.4.2** *Find the eigenvalues and an orthonormal basis of eigenvectors for* $A$.

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 4 & -2 \\ -2 & -2 & 7 \end{bmatrix}$$

Exercice 7.2.1

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$$\begin{bmatrix} 5 & -18 & -32 \\ 0 & 5 & 4 \\ 2 & -5 & -11 \end{bmatrix} \quad \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$\det(\lambda I - A) = 0$

$$\begin{bmatrix} \lambda-5 & 18 & 32 \\ 0 & \lambda-5 & -4 \\ -2 & 5 & \lambda+11 \end{bmatrix}$$

$$\det\begin{pmatrix} \lambda-5 & 18 & 32 \\ 0 & \lambda-5 & -4 \\ -2 & 5 & \lambda+11 \end{pmatrix} = 0$$

$$(\lambda-5)\left[(\lambda-5)(\lambda+11)+20\right] - 18(-8) + 32(+2\lambda-10) = 0$$

$$(\lambda-5)(\lambda^2+6\lambda-55+20) + 144 + 64\lambda - 320 = 0$$

$$\lambda^3+6\lambda^2-55\lambda+20\lambda-5\lambda^2-30\lambda+275-100+144+64\lambda-320 = 0$$

$$\lambda^3+\lambda^2-\lambda-1 = 0$$

$$(\lambda-1)(\lambda+1)(\lambda+1) = 0$$

$\lambda_1 = 1 \quad \lambda_2 = -1 \quad \lambda_3 = -1$

$\lambda_1 = 1 \quad \lambda_2 = \lambda_3 = -1$

for $\lambda_1 = 1$

$(\lambda I - A)x = 0$

$$\begin{bmatrix} -4 & 18 & 32 \\ 0 & -4 & -4 \\ -2 & 5 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -4 & 18 & 32 & 0 \\ 0 & -4 & -4 & 0 \\ -2 & 5 & 12 & 0 \end{array}\right] = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$2R_3 - R_1 \to R_3$

$$\left[\begin{array}{ccc|c} -4 & 18 & 32 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & -8 & -8 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -2 & 9 & 16 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 - 9R_2 \to R_1$

$$\left[\begin{array}{ccc|c} -2 & 0 & 7 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

infinite solutions

$z = t$ $\quad -2x + 7z = 0$

$y + z = 0$ $\quad -2x = -7z$

$y = -t$ $\quad x = \frac{7}{2}z = \frac{7}{2}t$

$$X_1 = \begin{bmatrix} 7/2 \\ -1 \\ 1 \end{bmatrix} t$$

$\lambda_2 = -1 \quad (-I - A)x = 0$

$$\begin{bmatrix} -6 & 18 & 32 \\ 0 & -6 & -4 \\ -2 & 5 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -6 & 18 & 32 & 0 \\ 0 & -6 & -4 & 0 \\ -2 & 5 & 10 & 0 \end{array}\right] = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$3R_3 - R_1 \to R_3$

$$\left[\begin{array}{ccc|c} -6 & 18 & 32 & 0 \\ 0 & -6 & -4 & 0 \\ 0 & -3 & -2 & 0 \end{array}\right]$$

$2R_3 - R_2 \to R_3$

$$\left[\begin{array}{ccc|c} -6 & 18 & 32 & 0 \\ 0 & -6 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad X_2 = \begin{bmatrix} 10/3 \\ -2/3 \\ 1 \end{bmatrix} t$$

$3R_2 + R_1 \to R_1$

$$\begin{bmatrix} -6 & 0 & 20 \\ 0 & -6 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$z = t$ $\quad -6x + 20(\;)t = 0$

$y = -\frac{2}{3}t$ $\quad x = \frac{20}{6}t = \frac{10}{3}t$

Exercice 7.2.2.
$$\begin{bmatrix} -13 & -28 & 28 \\ 4 & 9 & -8 \\ -4 & -8 & 9 \end{bmatrix}$$

$$\det(xI - A) = 0$$

$$\det\left( x \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -13 & -28 & 28 \\ 4 & 9 & 8 \\ -4 & -8 & 9 \end{bmatrix} \right) = 0$$

$$\det \begin{bmatrix} x+13 & 28 & -28 \\ -4 & x-9 & 8 \\ 4 & 8 & x-9 \end{bmatrix} = 0$$

$$4 \begin{bmatrix} 28 & -28 \\ x-9 & 8 \end{bmatrix} - 8 \begin{bmatrix} x+13 & -28 \\ -4 & 8 \end{bmatrix} + (x-9) \begin{bmatrix} x+13 & 28 \\ -4 & x-9 \end{bmatrix}$$

$$= 4(224 + 28x - 252) - 8(8x + 104 - 112) + (x-9)\left[(x+13)(x-9) + 112\right]$$

$$= 4(28x - 28) - 8(8x - 8) + (x-9)\left[x^2 - 9x + 13x - 117 + 112\right]$$

$$= 112(x-1) - 64(x-1) + (x-9)(x^2 + 4x - 5)$$

$$= 112(x-1) - 64(x-1) + (x-9)(x-1)(x+5)$$
$$= (x-1)\left[112 - 64 + (x-9)(x+5)\right]$$

$= (x-1)\left[48 + x^2 + 5x - 9x - 45\right]$

$= (x-1)(x^2 - 4x + 3) = (x-1)(x-3)(x-1) = 0$

$x = 1 \; ; \; x = 1 \; ; \; x = 3$ egenvalues

$(xI - A) = 0 \; ; \; x = 1$

$$\left(1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -13 & -28 & 28 \\ 4 & 9 & -8 \\ -4 & -8 & 9 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 28 & -28 \\ -4 & -8 & 8 \\ 4 & 8 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$7R_2 + 2R_1$

$7R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 14 & 28 & -28 & 0 \\ -4 & -8 & 8 & 0 \\ 4 & 8 & -8 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 14 & 28 & -28 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x = -2y + 2z$

$y = s$

$z = t$

$$\begin{bmatrix} -2s+2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} t$$

$(xI - A) = 0 ;\ x = 3$

$$\left( 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -13 & -28 & 28 \\ 4 & 9 & -8 \\ -4 & -8 & 9 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 16 & 28 & -28 \\ -4 & -6 & 8 \\ 4 & 8 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4R_2 + R_1$
$4R_3 - R_1$

$$\left[\begin{array}{ccc|c} 16 & 28 & -28 & 0 \\ -4 & -6 & 8 & 0 \\ 4 & 8 & -6 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 16 & 28 & -28 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right]$$

$R_3 - R_2$

$$\left[\begin{array}{ccc|c} 16 & 28 & -28 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad z = t$$

$$\begin{bmatrix} 7/2\, t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 7/2 \\ -1 \\ 1 \end{bmatrix} t$$

egen vector: $\begin{bmatrix} 7/2 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Exercice 7.2.3

3) $\begin{bmatrix} 89 & 38 & 268 \\ 14 & 2 & 40 \\ -30 & -12 & -90 \end{bmatrix}$

$\det(\lambda I - A) = 0$

$$\begin{bmatrix} x-89 & -38 & -268 \\ -14 & x-2 & -40 \\ 30 & 12 & x+90 \end{bmatrix} = 0$$

- $(x-89)\left[(x-2)(x+90) + (12)(40)\right] + 38\left[-14(x+90) + 1200\right]$

$-268\left[-14(12) - 30(x-2)\right] = 0$

- $(x-89)(x^2 + 90x - 2x - 180 + 480) + 38(-14x - 1260 + 1200)$
$-268(-168 - 30x + 60) = 0$

- $(x-89)(x^2 + 88x + 300) + 38(-14x - 60) - 268(-30x - 108) = 0$

- $x^3 + 88x^2 + 300x - 89x^2 - 7832x - 26700$
$-532x - 2280 + 8040x + 28944 = 0$

- $x^3 - x^2 - 24x - 36 = 0$

$x_1 = 6 \quad x_2 = -2 \quad x_3 = -3$

$\lambda_1 = 6 \quad \lambda_2 = -2 \quad \lambda_3 = -3$

$\det(xI - A) = 0$

$$\det\left( 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 89 & 38 & 268 \\ 14 & 2 & 40 \\ -30 & -12 & -90 \end{bmatrix} \right) = 0$$

$$\begin{bmatrix} -83 & -38 & -268 \\ -14 & 4 & -40 \\ 30 & 12 & 96 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -83 & -38 & -268 & 0 \\ -14 & 4 & -40 & 0 \\ 30 & 12 & 96 & 0 \end{array}\right]$$

$83R_2 - 14R_1$

$83R_3 + 30R_1$

$$\left[\begin{array}{ccc|c} -83 & -38 & -268 & 0 \\ 0 & 864 & 432 & 0 \\ 0 & -144 & -72 & 0 \end{array}\right]$$

Row eshlonform

$6R_3 + R_2$

$$\left[\begin{array}{ccc|c} -83 & -38 & -268 & 0 \\ 0 & 264 & 432 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = t$

$y = -\frac{1}{2}t$

$x = -3t$

[redacted]

$\det(-2I - A) = 0$

$$\det\left( 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 89 & 38 & 268 \\ 14 & 2 & 40 \\ -30 & -12 & -90 \end{bmatrix} \right) = 0$$

$$\begin{bmatrix} -91 & -38 & -268 \\ -14 & -4 & -40 \\ 30 & 12 & 88 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$13R_2 - 2R_1$

$91R_3 + 30R_1$

$$\left[\begin{array}{ccc|c} -91 & -38 & -268 & 0 \\ -14 & -4 & -40 & 0 \\ 30 & 12 & 88 & 0 \end{array}\right]$$

$2R_2 + R_3$

$$\left[\begin{array}{ccc|c} -91 & -38 & -268 & 0 \\ 0 & 24 & 16 & 0 \\ 0 & -48 & -32 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -91 & -38 & -268 & 0 \\ 0 & 12 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$12y = -8t \Rightarrow y = \frac{-8}{12}t$

$z = t$

$-91x = 38\left(\frac{-8}{12}\right)t + 268t$

$\Rightarrow x = \frac{-8}{3}t$

$$\begin{bmatrix} -\frac{8}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

$$\det(-3I - A) = 0$$

$$\det\left(-3\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \begin{bmatrix}89 & 38 & 268\\14 & 2 & 40\\-30 & -12 & -90\end{bmatrix}\right) = 0$$

$$\begin{bmatrix}-92 & -38 & -268\\-14 & -5 & -40\\30 & 12 & 87\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}-92 & -38 & -268 & 0\\-14 & -5 & -40 & 0\\30 & 12 & 87 & 0\end{array}\right] \quad \begin{array}{l}46R_2 - 7R_1\\46R_3 + 15R_1\end{array}$$

$$2R_3 + R_2 \quad \left[\begin{array}{ccc|c}-92 & -38 & -268 & 0\\0 & 36 & 36 & 0\\0 & -18 & -18 & 0\end{array}\right]$$

$$\left[\begin{array}{ccc|c}-92 & -38 & -268 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right] \quad \begin{array}{l}z = t\\y = -t\\x = -\frac{5}{2}\end{array}$$

$$\begin{bmatrix}-\frac{5}{2}\\-1\\1\end{bmatrix}$$

Exercice 7.2.4

$\det(xI-A)=0.$

$$\det\begin{bmatrix} x-1 & -90 & 0 \\ 0 & x+2 & 0 \\ -3 & -89 & x+2 \end{bmatrix} = 0$$

$(x-1)(x+2)^2 + 90(0) + 0 = 0.$

$(x-1)(x+2)^2 = 0$

$\lambda_1 = 1 \qquad \lambda_2 = -2 \qquad \lambda_3 = -2$ ( $-2$ is a root of multiplicity 2. )

for $\lambda_1 = 1$: $AX = X$

$(I-A)X = 0.$

$$\begin{bmatrix} 0 & -90 & 0 \\ 0 & 3 & 0 \\ -3 & -89 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} 0 & -90 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ -3 & -89 & 3 & 0 \end{array}\right]$$ Switch row 1 and row 3.

$$\left[\begin{array}{ccc|c} -3 & -89 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -90 & 0 & 0 \end{array}\right]$$

$30R_2 + R_3 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} -3 & -89 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ Divide 1st row by $(-3)$. Divide 2nd row by 3

$$\left[\begin{array}{ccc|c} 1 & \frac{89}{3} & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 - \frac{89}{3}R_2 \leftrightarrow R_1$ $$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ $\boxed{z=t}$ $\boxed{y=0}$

$x - z = 0 \Rightarrow \boxed{x=t}$

solution:

$$t\begin{bmatrix}1\\0\\1\end{bmatrix} \Rightarrow x_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

for $\lambda_2 = \lambda_3 = -2$ :

$$(-2I - A)X = 0.$$

$$\begin{bmatrix}-3 & -90 & 0\\0 & 0 & 0\\-3 & -89 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c}-3 & -90 & 0 & 0\\0 & 0 & 0 & 0\\-3 & -89 & 0 & 0\end{array}\right]$$

Switch rows 2 and 3.

$$\left[\begin{array}{ccc|c}-3 & -90 & 0 & 0\\-3 & -89 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

$R_2 - R_1 \leftrightarrow R_2$.

$$\left[\begin{array}{ccc|c}-3 & -90 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

$z = t$

$y = 0$

$-3x - 90y = 0 \Rightarrow x = 0.$

$$X_2 = X_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

A is not diagonalizable since it has only 2 eigenvectors.

Exercice 7.2.5

Exercise 7.2.5:

$$\det(\lambda I - A) = 0.$$

$$\det \begin{bmatrix} x-11 & -45 & -30 \\ -10 & x-26 & -20 \\ 20 & 60 & x+44 \end{bmatrix} = 0.$$

$$(x-11)\left[(x-26)(x+44)+1200\right] + 45(-10x-440+400) - 30(-600-[20(x-26)]) = 0$$

$$(x-11)(x^2+44x-26x-1144+1200) + 45(-10x-40) - 30(-600-20x+520) = 0.$$

$$(x-11)(x^2+18x+56) - 450x - 1800 + 18000 + 600x - 15600 = 0$$

$$x^3+18x^2+56x-11x^2-198x-616+150x+600 = 0.$$

$$x^3+7x^2+8x-16 = 0.$$

$\lambda_1 = 1$ $\lambda_2 = -4$ $\lambda_3 = -4$ ($-4$ is a root of multiplicity 2).

for $\lambda_1 = 1$ $(I-A)X = 0$

$$\begin{bmatrix} -10 & -45 & -30 \\ -10 & -25 & -20 \\ 20 & 60 & 45 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} -10 & -45 & -30 & 0 \\ -10 & -25 & -20 & 0 \\ 20 & 60 & 45 & 0 \end{array}\right]$$

$R_2 - R_1 \leftrightarrow R_2$
$R_3 + 2R_1 \leftrightarrow R_3$.

$$\left[\begin{array}{ccc|c} -10 & -45 & -30 & 0 \\ 0 & 20 & 10 & 0 \\ 0 & -30 & -15 & 0 \end{array}\right]$$

$20R_3 + 30R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} -10 & -45 & -30 & 0 \\ 0 & 20 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$45\,R_2 + 20\,R_1 \leftrightarrow R_1$

$$\left[\begin{array}{ccc|c} -200 & 0 & -150 & 0 \\ 0 & 20 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Divide $1^{st}$ row by $(-200)$.
Divide $2^{nd}$ row by $20$.

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{4} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\boxed{z = t}$

$y + \frac{1}{2} z = 0 \Rightarrow \boxed{y = -\frac{1}{2} t}$

$x + \frac{3}{4} z = 0 \Rightarrow \boxed{x = -\frac{3}{4} t}$

Solution

$$t \begin{bmatrix} -\frac{3}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \Rightarrow X_1 = \begin{bmatrix} -\frac{3}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

for $\lambda_2 = -4$: $(-4I - A)X = 0$

$$\begin{bmatrix} -15 & -45 & -30 \\ -10 & -30 & -20 \\ 20 & 60 & 40 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} -15 & -45 & -30 & 0 \\ -10 & -30 & -20 & 0 \\ 20 & 60 & 40 & 0 \end{array}\right]$$

$15\,R_2 - 10\,R_1 \leftrightarrow R_2$
$15\,R_3 + 20\,R_1 \leftrightarrow R_3$.

$$\left[\begin{array}{ccc|c} -15 & -45 & -30 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Divide $1^{st}$ row by $(-15)$.

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\boxed{z = t}$

$\boxed{y = s}$

$x + 3y + 2z = 0 \Rightarrow \boxed{x = -3s - 2t}$

Solution:

$$t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \qquad X_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$P = [\, X_1 \quad X_2 \quad X_3 \,]$$

$$P = \begin{bmatrix} -\frac{3}{4} & -3 & -2 \\ -\frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -4 & -12 & -8 \\ -2 & -5 & -4 \\ 4 & 12 & 9 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -4 & -12 & -5 \\ -2 & -5 & -4 \\ 4 & 12 & 9 \end{bmatrix} \begin{bmatrix} 11 & 45 & 30 \\ 10 & 26 & 20 \\ -20 & -60 & -44 \end{bmatrix} \begin{bmatrix} -\frac{3}{4} & -3 & -2 \\ -\frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{array}{c} \lambda_1 \\ \\ \\ \end{array} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}}_{D:\ \text{diagonal matrix}} \begin{array}{c} \\ \lambda_2 \\ \lambda_3 \end{array}$$

Exercice 7.2.6

$$\begin{bmatrix} 95 & 25 & 24 \\ -196 & -53 & -48 \\ -164 & -42 & -43 \end{bmatrix}$$

$\det(\lambda I - A) = 0$

$$\begin{vmatrix} x-95 & -25 & -24 \\ 196 & x+53 & 48 \\ 164 & 42 & x+43 \end{vmatrix} = 0$$

$(x-95)[(x+53)(x+43) - 2016]$
$+25(196(x+43) - 7872)$
$-24(8232 - 164x - 8692) = 0$

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$(x-95)(x^2 + 96x + 2279 - 2016)$
$+25(196x + 8428 - 7872)$
$-24(-164x - 460) = 0$

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$(x-95)(x^2 + 96x + 263)$
$+25(196x + 556) + 3936x + 11040 = 0$

$x^3 + 96x^2 + 263x - 95x^2 - 9120x - 24985$
$+ 4900x + 13900 + 3936x + 11040 = 0$

$x^3 + x^2 - 21x - 45 = 0$

$x_1 = 5 \qquad x_2 = -3 \qquad x_3 = -3$

$\lambda_1 = 5 \qquad \lambda_2 = \lambda_3 = -3$

$(5I - A)x = 0$

$$\left[\begin{array}{ccc|c} -90 & -25 & -24 & 0 \\ 196 & 58 & 48 & 0 \\ 164 & 42 & 48 & 0 \end{array}\right]$$

$90R_2 + 196R_1 \rightarrow R_2 \quad 90R_3 + \overset{164}{\cancel{196}}R_1 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} -90 & -25 & -24 & 0 \\ 0 & 320 & -384 & 0 \\ 0 & -320 & 384 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 90 & -25 & -24 & 0 \\ 0 & 1 & -\frac{6}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 + 25R_2 \rightarrow R_1$

$$\left[\begin{array}{ccc|c} -90 & 0 & -54 & 0 \\ 0 & 1 & -6/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = t$

$y = \frac{6}{5}t$

$-90x = 54t$

$x = -\frac{3}{5}t$

$$x_1 = \begin{bmatrix} -3/5 \\ 6/5 \\ 1 \end{bmatrix}$$

$\lambda_2 = -3$

$$\left[\begin{array}{ccc|c} -98 & -25 & -24 & 0 \\ 196 & 50 & 48 & 0 \\ 164 & 42 & 40 & 0 \end{array}\right]$$

$R_2 + 2R_1 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} -98 & -25 & -24 & 0 \\ 0 & 0 & 0 & 0 \\ 164 & 42 & 40 & 0 \end{array}\right]$$

$98R_3 + 164R_1 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} -98 & -25 & -24 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \cancel{2840}\ 16 & -16 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -98 & -25 & -24 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -98 & 0 & -49 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = t$

$y = z = t$

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$2x = -z$

$x = -\frac{t}{2}$

$X_2 = \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$

ercise 7.3.1

$\det(xI - A) = 0.$

$\det\left(x\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\right) = 0.$

$\det\begin{bmatrix}x-1 & -2\\ -2 & x-1\end{bmatrix} = 0.$

$(x-1)(x-1) - (-2(-2)) = 0$

$x^2 - 2x + 1 - 4 = 0.$

$x^2 - 2x - 3 = 0.$

$\lambda_1 = 3 \qquad \lambda_2 = -1$

for $\lambda_1 = 3$ : $AX = 3X$

$(3I - A)X = 0$

$\begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = 0$

Augmented Matrix: $\left[\begin{array}{cc|c}2 & -2 & 0\\ -2 & 2 & 0\end{array}\right]$

$R_2 + R_1 \leftrightarrow R_2$

$\left[\begin{array}{cc|c}2 & -2 & 0\\ 0 & 0 & 0.\end{array}\right]$ Divide $1^{st}$ row by $(2)$

$\left[\begin{array}{cc|c}1 & -1 & 0\\ 0 & 0 & 0\end{array}\right]$

$\boxed{y = t}$

Solution:

$x - y = 0 \implies \boxed{x = t}$

$t\begin{bmatrix}1\\ 1\end{bmatrix} \implies X_1 = \begin{bmatrix}1\\ 1\end{bmatrix}$

for $\lambda_2 = -1$ : $AX = -X$

$$(-I - A)X = 0$$

$$\left(-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Augmented matrix: $\left[\begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array}\right]$

$R_2 - R_1 \longleftrightarrow R_2$ $\left[\begin{array}{cc|c} -2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$$\boxed{y = s}$$

$$-2x - 2y = 0$$

$$-2x = 2s \Rightarrow \boxed{x = -s}$$

Solution:

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} s \Rightarrow X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$P = [X_1 \quad X_2]$$

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Augmented Matrix: $\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$

$R_2 - R_1 \longleftrightarrow R_2$

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right]$$

$2R_1 + R_2 \longleftrightarrow R_1$

$$\left[\begin{array}{cc|cc} 2 & 0 & 1 & 1 \\ 0 & 2 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 2 & 0 & 1 & 1 \\ 0 & 2 & -1 & 1 \end{array}\right]$$ Divide $1^{st}$ row and $2^{nd}$ row by 2

$$\underbrace{\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]}_{P^{-1}}$$

Then

$$P^{-1}AP$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\lambda_1 \leftarrow \quad = \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}}_{\text{diagonal matrix: } D} \quad \rightarrow \lambda_2$$

$$A = PDP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Therefore

$$A^{10} = PD^{10}P^{-1}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^{10} & 0 \\ 0 & (-1)^{10} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} 3^{10} & -1 \\ 3^{10} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \quad A^{10} = \begin{bmatrix} \frac{3^{10}+1}{2} & \frac{3^{10}-1}{2} \\ \frac{3^{10}-1}{2} & \frac{3^{10}+1}{2} \end{bmatrix}$$

Exercise 7.3.2

Ex : 7.3.2 $A = \begin{pmatrix} 1 & 4 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 5 \end{pmatrix}$

• eigen values: $\det(xI - A) = 0$

$$\begin{vmatrix} x-1 & -4 & -1 \\ 0 & x-2 & -5 \\ 0 & 0 & x-5 \end{vmatrix} = 0 \Rightarrow (x-1)\big((x-2)(x-5)\big) = 0.$$

$\lambda_1 = 5$ ; $\lambda_2 = 2$ ; $\lambda_3 = 1$

• for $\lambda_1 = 5$ $\quad (\lambda_1 I - A)(X) = 0$

$$\left(\begin{array}{ccc|c} 4 & -4 & -1 & 0 \\ 0 & 3 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & -\frac{1}{4} & 0 \\ 0 & 1 & -5/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

• $\underline{\underline{z = t}} \Rightarrow y = \underline{\underline{\frac{5}{3}t}}$ ; $x - \frac{5}{3}t - \frac{1}{4}t = 0 \Rightarrow x = \underline{\underline{\frac{23t}{12}}}$

$$X_1 = \begin{pmatrix} 23/12 \\ 5/3 \\ 1 \end{pmatrix}$$

• for $\lambda_2 = 2$ $\left(\begin{array}{ccc|c} 1 & -4 & -1 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -4 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$z = \underline{\underline{0}}$ $\quad y = \underline{\underline{t}}$ $\quad x - 4t = 0 \Rightarrow x = \underline{\underline{4t}}$

$$X_2 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

for $\lambda_3 = 1$

$$\left(\begin{array}{ccc|c} 0 & -4 & -1 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right) R_2 \to 4R_2 - R_1 \longrightarrow$$

$$\left(\begin{array}{ccc|c} 0 & -4 & -1 & 0 \\ 0 & 0 & -19 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 0 & 1 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So $\quad P = \begin{pmatrix} 23/12 & 4 & 1 \\ 5/3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$P^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -5/3 \\ 1 & -4 & \frac{19}{4} \end{pmatrix}$$

$$D = P^{-1}AP = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -5/3 \\ 1 & -4 & \frac{19}{4} \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{23}{12} & 4 & 1 \\ 5/3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{50} = PD^{50}P^{-1} = \begin{pmatrix} \frac{23}{12} & 4 & 1 \\ 5/3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5^{50} & 0 & 0 \\ 0 & 2^{50} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -\frac{5}{3} \\ 1 & -4 & \frac{19}{4} \end{pmatrix}$$

Exercise 7.3.3

Ex 7.3.3 $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -2 & 3 & 1 \end{bmatrix}$

.eigen values:

$\det(xI - A) = 0 \Rightarrow \begin{vmatrix} x-1 & 2 & 1 \\ -2 & x+1 & -1 \\ 2 & -3 & x-1 \end{vmatrix} = 0$

$\Rightarrow (x-1)\left[(x+1)(x-1) - 3\right] - 2\left[(-2)(x-1) + 2\right] + \left[6 - 2(x+1)\right] = 0$

$(x-1)(x^2 - 1 - 3) - 2(-2x + 2 + 2) + (6 - 2x - 2) = 0$

$\Rightarrow (x-1)(x^2 - 4) - 2(-2x + 4) + (-2x + 4) = 0$

$\Rightarrow x^3 - \cancel{4x} - x^2 + \cancel{4} + \cancel{4x} - 8 - 2x + \cancel{4} = 0$

$\Rightarrow x^3 - x^2 - 2x = 0$ $\quad \underline{\underline{\lambda_1 = 2}}$ $\quad \underline{\underline{\lambda_2 = -1}}$ $\quad \underline{\underline{\lambda_3 = 0}}$

* for $\lambda_1 = 2$ . $(2I - A)(X) = 0$

$\Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ -2 & 3 & -1 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -2 & 3 & -1 & 0 \\ 2 & -3 & 1 & 0 \end{array}\right) \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \to$

$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & -7 & -1 & 0 \end{array}\right) \xrightarrow{R_3 \to R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\cdot\ \underline{\underline{z = t}} \Rightarrow y = -\frac{1}{7}t \Rightarrow x - \frac{2}{7}t + t = 0 \Rightarrow \underline{\underline{x = -\frac{5}{7}t}}$

$X_1 = \begin{pmatrix} -\frac{5}{7} \\ -\frac{1}{7} \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -5 \\ -1 \\ 7 \end{pmatrix}}}$

for $\lambda_2 = -1$ $\quad (-I - A)X = 0$

$$\Rightarrow \left(\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ -2 & 0 & -1 & 0 \\ 2 & -3 & -2 & 0 \end{array}\right) \begin{array}{l} R_2 \to R_1 - R_2 \\ R_3 \to R_1 + R_3 \end{array} \longrightarrow$$

$$\left(\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right) \xrightarrow{R_3 \to R_2 - 2R_3} \left(\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \underline{\underline{So}} \quad z = \underline{\underline{t}}, \; y = -\underline{\underline{t}} \;\&\; x = -\underline{\underline{\tfrac{1}{2}t}}$$

$$X_2 = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{pmatrix}^{\times 2} = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$$

for $\lambda_3 = 0$ $\quad (0I - A)(X) = 0$

$$\left(\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ -2 & 1 & -1 & 0 \\ 2 & -3 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \underline{\underline{So}} \quad z = \underline{\underline{t}} \quad y = -\underline{\underline{t}} \;\&\; x = -\underline{\underline{t}}$$

$$X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} -5 & -1 & -1 \\ 1 & -2 & -1 \\ 7 & 2 & 1 \end{pmatrix}$$

$\ast$ $P^{-1} = ?$

$\cdot \det(P) = -6 \qquad \cdot \operatorname{Cof}(P) = \begin{pmatrix} 0 & -6 & 12 \\ -1 & 2 & 3 \\ 1 & -4 & 9 \end{pmatrix}$

$$P^{-1} = \begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{6} \\ 1 & -\frac{1}{3} & \frac{2}{3} \\ -2 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}$$

$$D = P^{-1}AP = \begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{6} \\ 1 & -\frac{1}{3} & \frac{2}{3} \\ -2 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} -5 & -1 & -1 \\ -1 & -2 & -1 \\ 7 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow A^{100} = PD^{100}P^{-1} = \begin{pmatrix} -5 & -1 & -1 \\ 1 & -2 & -1 \\ 7 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2^{100} & 0 & 0 \\ 0 & -1^{100} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{6} \\ 1 & -\frac{1}{3} & \frac{2}{3} \\ -2 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} -1 & -\frac{5}{6} \times 2^{100} + \frac{1}{3} & -\frac{5}{6} \times 2^{100} - \frac{2}{3} \\ -2 & -\frac{1}{6} \times 2^{100} + \frac{2}{3} & -\frac{1}{6} \times 2^{100} - \frac{4}{3} \\ 2 & \frac{7}{6} \times 2^{100} - \frac{2}{3} & \frac{7}{6} \times 2^{100} + \frac{4}{3} \end{pmatrix}$$

Exercise 7.4.1

# 7.4.1:

$\det (xI - A) = 0.$

$$\det \begin{bmatrix} x-11 & 1 & 4 \\ 1 & x-11 & 4 \\ 4 & 4 & x-14 \end{bmatrix} = 0$$

$(x-11)[(x-11)(x-14)-16] - [x-14-16] + 4[4-(4(x-11))] = 0.$

$(x-11)(x^2-14x-11x+154-16) - x + 30 + 4(4-4x+44) = 0.$

$(x-11)(x^2-25x+138) - x + 30 + 4(-4x+48) = 0.$

$x^3 - 25x^2 + 138x - 11x^2 + 275x - 1518 - x + 30 - 16x + 192 = 0.$

$x^3 - 36x^2 + 396x - 1296 = 0.$

$\lambda_1 = 6. \quad \lambda_2 = 18 \quad \lambda_3 = 12$

For $\lambda_1 = 6$: $(6I - A)X = 0.$

$$\begin{bmatrix} -5 & 1 & 4 \\ 1 & -5 & 4 \\ 4 & 4 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} -5 & 1 & 4 & 0 \\ 1 & -5 & 4 & 0 \\ 4 & 4 & -8 & 0 \end{array}\right]$$

$5R_2 + R_1 \longleftrightarrow R_2$
$5R_3 + 4R_1 \longleftrightarrow R_3$

$$\left[\begin{array}{ccc|c} -5 & 1 & 4 & 0 \\ 0 & -24 & 24 & 0 \\ 0 & 24 & -24 & 0 \end{array}\right]$$

$R_3 + R_2 \longleftrightarrow R_3.$

$$\left[\begin{array}{ccc|c} -5 & 1 & 4 & 0 \\ 0 & -24 & 24 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$24\ R_1 + R_2 \leftrightarrow R_1$

$$\left[\begin{array}{ccc|c} -120 & 0 & 120 & 0 \\ 0 & -24 & 24 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ Divide 1st row by $(-120)$

Divide 2nd row by $(-24)$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{z = t}$$

$$y - z = 0 \Rightarrow \boxed{y = t}$$

$$x - z = 0 \Rightarrow \boxed{x = t}$$

eigenvector: $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

<u>for $\lambda_2 = 18$</u>: $(18I - A)X = 0$

$$\begin{bmatrix} 7 & 1 & 4 \\ 1 & 7 & 4 \\ 4 & 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} 7 & 1 & 4 & 0 \\ 1 & 7 & 4 & 0 \\ 4 & 4 & 4 & 0 \end{array}\right]$$

$7R_2 - R_1 \leftrightarrow R_2$

$7R_3 - 4R_1 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 7 & 1 & 4 & 0 \\ 0 & 48 & 24 & 0 \\ 0 & 24 & 12 & 0 \end{array}\right]$$

$$2R_3 - R_2 \longleftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 7 & 1 & 4 & 0 \\ 0 & 48 & 24 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$48 R_1 - R_2 \longleftrightarrow R_1$$

$$\left[\begin{array}{ccc|c} 336 & 0 & 168 & 0 \\ 0 & 48 & 24 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Divide $1^{st}$ row by 336

Divide $2^{nd}$ row by 48

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{z = t}$$

$$y + \frac{1}{2} z = 0 \Rightarrow \boxed{y = -\frac{1}{2} t}$$

$$x + \frac{1}{2} z = 0 \Rightarrow \boxed{x = -\frac{1}{2} t}$$

$$x_2 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

for $\lambda_3 = 12$, $(12I - A)X = 0$

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 4 \\ 4 & 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 1 & 1 & 4 & 0 \\ 4 & 4 & -2 & 0 \end{array}\right]$$

$R_2 - R_1 \leftrightarrow R_2$

$R_3 - 4R_1 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -18 & 0 \end{array}\right]$$

Divide $3^{rd}$ row by $(-18)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\boxed{y = t}$

$\boxed{z = 0}$

$x + y + 4z = 0 \Rightarrow \boxed{x = -t}$

$$X_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{bmatrix}$$

$$P^{T}AP = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 11 & -1 & -4 \\ -1 & 11 & -4 \\ -4 & -4 & 14 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2\sqrt{3} & 2\sqrt{3} & 2\sqrt{3} \\ -3\sqrt{6} & -3\sqrt{6} & 6\sqrt{6} \\ -6\sqrt{2} & 6\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{bmatrix}$$

$$= \begin{array}{ccc} x_1 \leftarrow & \begin{bmatrix} 6 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 12 \end{bmatrix} & \begin{array}{l} \\ \rightarrow x_2 \\ \rightarrow x_3. \end{array} \end{array}$$

which is the desired diagonal matrix

Exercice 7.4.2

7.4.2.

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 4 & -2 \\ -2 & -2 & 7 \end{bmatrix} \quad \lambda I = \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}$$

$\det(\lambda I - A) = 0$

$$\begin{vmatrix} x-4 & -1 & 2 \\ -1 & x-4 & 2 \\ 2 & 2 & x-7 \end{vmatrix} = 0$$

$(x-4)\left[(x-4)(x-7)-4\right] + \left[(-x+7)-4\right] + 2(-2-2x+8) = 0$

$(x-4)(x^2-11x+28-4) - x + 3 + 12 - 4x = 0$

$x^3 - 11x^2 + 24x - 4x^2 + 44x - 96 - 5x + 15 = 0$

$x^3 - 15x^2 + 63x - 81 = 0$

---

$x_1 = 9 \qquad x_2 = 3 \qquad x_3 = 3$

$\lambda_1 = 9 \qquad \lambda_2 = 3 \qquad \lambda_3 = 3$

---

$\lambda_1 = 9$

$(9I - A)x = 0$

$$\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 5 & 2 & 0 \\ 5 & -1 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & -6 & -3 & 0 \end{array}\right]$$

$\cdot\ R_2 + R_1 \rightarrow R_2$

$\cdot\ R_3 - 5R_1 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad X_1 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

divide by $\sqrt{\frac{3}{2}}$

$z = t$

$6y + 3z = 0$

$y = -\frac{1}{2}t$

$x - \frac{1}{2}t + t = 0$

$x = -\frac{1}{2}t$

$$X_1 = \begin{bmatrix} -1/2 \,/\, \sqrt{3/2} \\ -\frac{1}{2} \,/\, \sqrt{3/2} \\ 1 \,/\, \sqrt{\frac{3}{2}} \end{bmatrix}$$

$$\rightarrow X_1 = \begin{bmatrix} -\frac{\sqrt{2}}{2\sqrt{3}} \\ -\frac{\sqrt{2}}{2\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{bmatrix}$$

$\underline{\lambda_2 = 3}$

$$\left[\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ -1 & -1 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 2 & -4 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 - R_2 \rightarrow R_3$
$2R_2 + R_1 \rightarrow R_2$

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$z = t$
$y = s$
$2x + 2s - 4t = 0$
$x = 2t - s$

$$\begin{bmatrix} 2t - s \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s$$

divide $\sqrt{5}$ divide $\sqrt{2}$

$$\lambda_2 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix} \quad \lambda_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} -\sqrt{2}/2\sqrt{3} & 2/\sqrt{5} & -1/\sqrt{2} \\ -\sqrt{2}/2\sqrt{3} & 0 & 1/\sqrt{2} \\ \sqrt{2/3} & 1/\sqrt{5} & 0 \end{bmatrix} \quad P^T = \begin{bmatrix} -\frac{\sqrt{2}}{2\sqrt{3}} & -\frac{\sqrt{2}}{2\sqrt{3}} & \sqrt{\frac{2}{3}} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$